Differential Geometry IV: General relativity

11.1 Let g be a Lorentzian metric on $\mathbb{R}^{3+1}$ with the property that, in the standard Cartesian coordinates (t, x^1, x^2, x^3) on $\mathbb{R}^{3+1}$, we have

$$\left|g_{\alpha\beta} - \eta_{\alpha\beta}\right| < \frac{1}{10}$$

and that the components of the deformation tensor $\pi^{(T)}_{\alpha\beta} = \nabla_\alpha T_\beta + \nabla_\beta T_\alpha$ of the vector field $T = \partial_t$ satisfy

$$|\pi^{(T)}_{\alpha\beta}| \leqslant \frac{1}{(1+|t|)^{1+\delta}}$$

for some $\delta > 0$. Show that, for any initial data set $(\psi_0, \psi_1) \in C^\infty_0(\mathbb{R}^3) \times C^\infty_0(\mathbb{R}^3)$ at $\{t = 0\}$, the corresponding solution ψ of the wave equation $\square_g \psi = 0$ satisifes

$$\sup_t \sum_{\alpha=0}^{3} \|\partial_\alpha \psi(t, \cdot)\|_{L^2(\mathbb{R}^3)} \leqslant C_\delta \Big(\|\psi_0\|_{H^1} + \|\psi_1\|_{L^2} \Big)$$

for some constant $C_\delta > 0$ depending only on δ.

(Hint: Apply the divergence identity for the vector field T.)

11.2 In this exercise, we will establish a number functional inequalities on $\mathbb{R}^3$ that are frequently used in analysis in order to control L^P norms of a given function by L^q norms of its higher order derivatives. To this end, for any function $f \in C^\infty_0(\mathbb{R}^3)$ and any $k \in \mathbb{N}$, we will set

$$\|\nabla^k_x f\|^2_{L^2_x(\mathbb{R}^3)} \doteq \sum_{\substack{(a_1,a_2,a_3)\in\mathbb{N}^3:\\ a_1+a_2+a_3=k}} \int_{\mathbb{R}^3} \Big| \frac{\partial^{a_1+a_2+a_3}}{(\partial x^1)^{a_1}(\partial x^2)^{a_2}(\partial x^3)^{a_3}} f \Big|^2 \, dx.$$

We will also define the Sobolev norm by

$$\|f\|^2_{H^k(\mathbb{R}^3)} \doteq \sum_{l=0}^{k} \|\nabla^l_x f\|^2_{L^2(\mathbb{R}^3)}$$

and, for the *homogeneous* Sobolev norm:

$$\|f\|_{\dot{H}^k(\mathbb{R}^3)} \doteq \|\nabla^k_x f\|_{L^2(\mathbb{R}^3)}$$

(note the dot in the notation for $\dot{H}^k$ vs H^k).

(a) Denoting with $\hat{f}$ the Fourier transform of f, i.e.

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{2\pi i \xi \cdot x} f(x) \, dx,$$

show that there exists a constant $C_k > 0$ depending only on $k \in \mathbb{N}$ such that

$$\frac{1}{C_k}\||\xi|^k \hat{f}\|_{L^2_\xi(\mathbb{R}^3)} \leqslant \|f\|_{\dot{H}^k(\mathbb{R}^3)} \leqslant C_k\||\xi|^k \hat{f}\|_{L^2_\xi(\mathbb{R}^3)}.$$

Moreover, show that

$$\|f\|_{L^\infty_x(\mathbb{R}^3)} \leqslant \|\hat{f}\|_{L^1_\xi(\mathbb{R}^3)}.$$

Hint: For the last inequality, use the formula for the inverse Fourier operator $f(x) = \int_{\mathbb{R}^3} e^{-2\pi i\xi\cdot x}\hat{f}(\xi)\,d\xi$.

(b) Show that there exists a constant $C > 0$ such that, for all functions $f \in C_0^\infty(\mathbb{R}^3)$, the following **Sobolev-type** inequality holds:

$$\|f\|^2_{L^\infty_x(\mathbb{R}^3)} \leqslant C \cdot \Big(\|f\|^2_{\dot{H}^2_x(\mathbb{R}^3)} + \|f\|^2_{\dot{H}^1_x(\mathbb{R}^3)}\Big)$$

Remark. Note that, contrary to the more standard Sobolev inequality, the right hand side above does not contain a zeroth order term in f. This inequality is useful in the case of wave-type equations, where the naturally conserved energy-type norms usually do not involve undifferentiated terms of the unknown functions.

(c) Let $\Omega \subset \mathbb{R}^3$ be a *bounded* domain with smooth boundary. We will assume as given the following corollary of the more general Rellich–Kondrachov theorem: For any such domain (in fact, any such domain in $\mathbb{R}^n$), the identity map $\mathrm{id} : H^1(\Omega) \to L^2(\Omega)$ is *compact*, i.e. the set of functions $\{f : \|f\|_{H^1(\Omega)} \leqslant 1\}$ is pre-compact with respect to the L^2 topology.

Show that there exists a constant C (depending only on the domain Ω, such that

$$\|f\|_{L^2(\Omega)} \leqslant C\|\nabla_x f\|_{L^2(\Omega)} \quad \text{for all } f \in C^\infty(\Omega) \text{ such that } \int_\Omega f\,dx = 0.$$

This is a variant of the **Poincare inequality**.

(Hint: Assume, for the sake of contradiction, that there exists a sequence of functions f_n *with* $\int_\Omega f_n = 0$ *such that* $\|f_n\|_{L^2(\Omega)} = 1$ *and* $\|\nabla_x f_n\|_{L^2(\Omega)} \xrightarrow{n\to\infty} 0$. *)*

11.3 We will now extend the functional inequalities of the previous exercise to more general 3-dimensional Riemannian manifolds. To this end, let $(\mathcal{M}^3, h)$ be a *compact Riemannian* manifold. We will set for any (k,l)-tensor field T on $\mathcal{M}$:

$$\|T\|^2_{L^2(\mathcal{M})} = \int_{\mathcal{M}} \|T\|^2_h\,dvol_h,$$

where

$$\|T\|^2_h \doteq \langle T, T\rangle_h = h_{i_1a_1}\dots h_{i_ka_k}h^{j_1b_1}\dots h^{j_lb_l}T^{i_1\dots i_k}_{j_1\dots j_l}T^{a_1\dots a_k}_{b_1\dots b_l}$$

and $dvol_h$ is the natural volume form associated to h so that, in any local coordinate system (x^1, x^2, x^3), $dvol_h = \sqrt{\det(h)}dx^1dx^2dx^3$.

(a) Show that there exists a constant $C > 0$ (depending on $(\mathcal{M}, h)$) such that, for any $f \in C^\infty(\mathcal{M})$, we have

$$\|f\|^2_{L^\infty(\mathcal{M})} \leqslant C \cdot \Big(\|\nabla^2 f\|^2_{L^2(\mathcal{M})} + \|\nabla f\|^2_{L^2(\mathcal{M})} + \|f\|^2_{L^2(\mathcal{M})} \Big).$$

Hint: Fix a finite collection of compact sets $\{\mathcal{K}_n\}_n$ such that each $\mathcal{K}_n$ lies strictly inside a coordinate chart and the open sets int$(\mathcal{K}_n)$ cover all of $\mathcal{M}$. If $\chi_n : \mathcal{M} \to [0, +\infty)$ are cutoff-functions such that $\sum_n \chi_n = 1$ and supp$(\chi_n) \subset \mathcal{K}_n$, use the previous part of the exercise for the function $\chi_n \cdot f$.

(b) Using the Poincare inequality, show that there exists a $C > 0$ such that, for any $f \in C^\infty(\mathcal{M})$:

$$\|f - \bar{f}\|^2_{L^\infty(\mathcal{M})} \leqslant C \cdot \Big(\|\nabla^2 f\|^2_{L^2(\mathcal{M})} + \|\nabla f\|^2_{L^2(\mathcal{M})} \Big),$$

where

$$\bar{f} \doteq \int_{\mathcal{M}} f \, \mathrm{dvol}_h.$$

11.4 In this exercise, we will use the functional inequalities established earlier to deduce a general boundedness statement for solutions to the wave equation on static spacetimes.

Let $(\mathcal{M}^3, h)$ be a compact Riemannian manifold and let us consider the (static) Lorentzian metric

$$g = -dt^2 + h$$

on $\tilde{\mathcal{M}} = \mathbb{R} \times \mathcal{M}$, where t is the projection on the first factor of $\mathbb{R} \times \mathcal{M}$. Let also T denote the Killing vector field corresponding to translations in the $\mathbb{R}$ factor (so that, in any local coordinate system of the form (t, x^1, x^2, x^3) on $\tilde{\mathcal{M}}$, where (x^1, x^2, x^3) are coordinates on $\mathcal{M}$, we have that $T = \frac{\partial}{\partial t}$). For any function $\psi \in C^\infty(\tilde{\mathcal{M}})$, we will set

$$\mathcal{E}[\psi](t) \doteq \int_{\{t\} \times \mathcal{M}} \Big(|T(\psi)|^2 + \|\bar{\nabla}\psi\|^2_h \Big) \, \mathrm{dvol}_h,$$

where $\bar{\nabla}$ denotes the connection of h (so that in (t, x^1, x^2, x^3) cooridnates as before, $\bar{\nabla}\psi = \partial_i \psi dx^i$).

Let ψ be a smooth solution of the wave equation $\square_g \psi = 0$ on $\tilde{\mathcal{M}}$. In this exercise, we will see how the energy estimates can be used to obtain estimates on the global-in-time behaviour of ψ.

(a) Show that, for any $t \in \mathbb{R}$,

$$\mathcal{E}[\psi](t) = \mathcal{E}[\psi](0).$$

(b) Show that $T\psi$ is a also a solution of the wave equation. Moreover, show that, for any $t \in \mathbb{R}$,

$$\int_{\{t\}\times\mathcal{M}} (\Delta_h \psi)^2 \, \mathrm{dvol}_h \leqslant \mathcal{E}[T\psi](t),$$

where Δ_h is the elliptic Laplace-Beltrami operator associated to h:

$$\Delta_h = h^{ab} \bar{\nabla}_a \bar{\nabla}_b \psi$$

(Hint: Use the wave equation for ψ to get an expression for $T^2\psi$.)

*(c) Show that there exists a constant $C > 0$ depending only on $(\mathcal{M}, h)$ such that

$$\int_{\{t\}\times\mathcal{M}} |\bar{\nabla}^2 \psi|^2 \, \mathrm{dvol}_h \leqslant C\Big(\mathcal{E}[T\psi](t) + \mathcal{E}[\psi](t)\Big)$$

Hint: Use integration by parts in the expression

$$\int_{\{t\}\times\mathcal{M}} (\Delta_h \psi)^2 \, dvol_h = \int_{\{t\}\times\mathcal{M}} h^{ab} h^{ij} \bar{\nabla}_a \bar{\nabla}_b \psi \bar{\nabla}_i \bar{\nabla}_j \psi \, dvol_h,$$

noting that

$$\int_{\{t\}\times\mathcal{M}} |\bar{\nabla}^2 \psi|^2 \, dvol_h = \int_{\{t\}\times\mathcal{M}} h^{ab} h^{ij} \bar{\nabla}_a \bar{\nabla}_i \psi \bar{\nabla}_b \bar{\nabla}_j \psi \, dvol_h.$$

(d) Show that there exists a constant $C > 0$ depending onl on $(\mathcal{M}, h)$ such that

$$\sup_t \|\psi - \bar{\psi}\|^2_{L^\infty(\{t\}\times\mathcal{M})} \leqslant C \cdot \Big(\mathcal{E}[\psi](0) + \mathcal{E}[T\psi](0)\Big),$$

where

$$\bar{\psi}(t) \doteq \int_{\{t\}\times\mathcal{M}} \psi \, \mathrm{dvol}_h.$$

(e) Show that $\bar{\psi}(t)$ satisfies $\frac{d^2}{dt^2}\bar{\psi} = 0$. Deduce that if $\int_{t=0} (T\psi) \, \mathrm{dvol}_h = 0$, then

$$\sup_{\bar{\mathcal{M}}} |\psi| < +\infty.$$

What if $\int_{t=0} (T\psi) \, \mathrm{dvol}_h \neq 0$?